

Can gravity distinguish between Dirac and Majorana neutrinos?

Dinesh Singh^{a,*}, Nader Mobed^{a,†} and Giorgio Papini^{a,b,c‡}

^a*Department of Physics, University of Regina, Regina, Saskatchewan, S4S 0A2, Canada*

^b*Prairie Particle Physics Institute, Regina, Saskatchewan, S4S 0A2, Canada and*

^c*International Institute for Advanced Scientific Studies, 89019 Vietri sul Mare (SA), Italy*

(Dated: July 6, 2006)

We show that spin-gravity interaction can distinguish between Dirac and Majorana neutrino wave packets propagating in a Lense-Thirring background. Using time-independent perturbation theory and gravitational phase to generate a perturbation Hamiltonian with spin-gravity coupling, we show that the associated matrix element for the Majorana neutrino differs significantly from its Dirac counterpart. This difference can be demonstrated through significant gravitational corrections to the neutrino oscillation length for a two-flavour system, as shown explicitly for SN1987A.

PACS numbers: 04.90.+e, 14.60.Pq, 04.80.+z, 97.60.Bw

Introduction.—An unresolved question in the Standard Model is whether neutrinos exist as Dirac or Majorana particles [1, 2], where the latter are regarded as the more natural candidates [3] to exist in Nature. This is because the Majorana neutrino, as its own antiparticle, has only half the degrees of freedom compared to the Dirac neutrino, and is deemed a more fundamental particle as a result. One known process to distinguish between the two types is neutrinoless double beta decay [1, 2], which only occurs for Majorana neutrinos. However, direct observation of this phenomenon is at best inconclusive. As well, Dirac and Majorana neutrinos are potentially distinguishable in magnetic fields, since they possess unique magnetic moment structures [1], though this feature becomes relevant only when the fields are extremely strong. A recent discovery [4, 5] has shown that neutrinos undergo flavour oscillations while propagating in vacuum, inferring the existence of neutrino rest masses. However, the current theory of neutrino oscillations cannot distinguish between a Dirac and Majorana neutrino because only its left-handed chiral projection is subject to this phenomenon [6], which is identical for both types.

Although gravitational effects are usually neglected in particle physics, the fact that neutrinos are electrically neutral provides an opportunity to study their long-range behaviour in response to curved space-time. For example, it is possible that a massive neutrino's helicity state can be flipped due to explicit coupling between its spin and the background gravitational field. If such an interaction exists, then we have means to probe the intrinsic nature of neutrinos due to gravity, including the possibility to differentiate between Dirac and Majorana particles in a meaningful way. By treating the two neutrino types as massive wave packets propagating in a Lense-Thirring (LT) background [7], it is shown below that the Dirac and Majorana matrix elements differ significantly, along with their respective oscillation lengths.

Dirac Hamiltonian in Curved Space-Time.—We begin with the four-dimensional covariant Dirac equation $[i\gamma^\mu(x)D_\mu - \frac{m}{\hbar}]\psi(x) = 0$ of -2 signature ($\mu = 0, 1, 2, 3$)

with neutrino mass m , with $G = c = 1$ units [8]. The curved space-time gamma matrices satisfying $\{\gamma^\mu(x), \gamma^\nu(x)\} = 2g^{\mu\nu}(x)$ are expressed in terms of Minkowski gamma matrices $\gamma^{\hat{\mu}}$ and orthonormal vierbeins $\{e^\mu_{\hat{\mu}}\}$, with $g_{\mu\nu} = \eta_{\hat{\alpha}\hat{\beta}} e^{\hat{\alpha}}_\mu e^{\hat{\beta}}_\nu$, $e^{\hat{\alpha}}_\mu e^\mu_{\hat{\beta}} = \delta^{\hat{\alpha}}_{\hat{\beta}}$, $e^\mu_{\hat{\alpha}} e^{\hat{\alpha}}_\nu = \delta^\mu_\nu$. As well, $\gamma^\mu(x) = e^\mu_{\hat{\mu}} \gamma^{\hat{\mu}}$ and $\{\gamma^{\hat{\mu}}, \gamma^{\hat{\nu}}\} = 2\eta^{\hat{\mu}\hat{\nu}}$. Then $D_\mu = \partial_\mu + i\Gamma_\mu$ is the covariant derivative operator in terms of the spin connection $\Gamma_\mu = -\frac{1}{4}\sigma^{\alpha\beta}(x)\Gamma_{\alpha\beta\mu} = -\frac{1}{4}\sigma^{\hat{\alpha}\hat{\beta}}\Gamma_{\hat{\alpha}\hat{\beta}\hat{\mu}} e^{\hat{\mu}}_\mu$, where $\sigma^{\hat{\alpha}\hat{\beta}} = \frac{i}{2}[\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}]$ and $\Gamma_{\hat{\alpha}\hat{\beta}\hat{\mu}}$ are Ricci rotation coefficients. The LT metric [7] for $x^\mu = (t, x, y, z)$ is

$$g = \left(1 - \frac{2M}{r}\right) dt \otimes dt - \left(1 + \frac{2M}{r}\right) (dx \otimes dx + dy \otimes dy + dz \otimes dz) + \frac{4}{5} \frac{M\Omega R^2}{r^3} \times [x(dy \otimes dt + dt \otimes dy) - y(dx \otimes dt + dt \otimes dx)], \quad (1)$$

with $M/r \ll 1$ and $M\Omega R^2/r^2 \ll 1$, where $r = \sqrt{x^2 + y^2 + z^2}$, M and R are the mass and radius of the gravitational source, and Ω is its rotational frequency. The corresponding Dirac Hamiltonian to leading order in M/r is

$$H_0 \approx \left(1 - \frac{2M}{r}\right) \boldsymbol{\alpha} \cdot \mathbf{p} + m \left(1 - \frac{M}{r}\right) \beta + i\hbar \frac{M}{2r^3} (\boldsymbol{\alpha} \cdot \mathbf{r}) + \frac{4}{5} \frac{M\Omega R^2}{r^3} \mathbf{L}^z + \frac{1}{5} \frac{\hbar M\Omega R^2}{r^3} \left[\frac{3z}{r^2} (\boldsymbol{\Sigma} \cdot \mathbf{r}) - \boldsymbol{\Sigma}^z \right], \quad (2)$$

where $\boldsymbol{\alpha}$ and β are the Dirac matrices, $\boldsymbol{\Sigma}^j = \sigma^{\hat{0}j}$ is the x^j -component of the spin angular momentum operator, and \mathbf{L}^z is the orbital angular momentum operator in the z -direction. In spherical co-ordinates, the field point is $\mathbf{r} = (r, \theta, \varphi)$, defined in relation to a cartesian co-ordinate frame expressed by x^μ . The energy eigenvalue for $H_0 |\psi_0\rangle = E_0^{(\pm)} |\psi_0\rangle$ is

$$E_0^{(\pm)} \approx \sqrt{(\hbar k_0)^2 + m^2} - \frac{2M}{r} (\hbar k_0) + \frac{4}{5} \frac{M\Omega R^2}{r^3} \left(L^z \pm \frac{\hbar}{2} \right), \quad (3)$$

where $p = \hbar k_0$ is the neutrino's momentum eigenvalue.

Gravitational Phase.—By itself, (2) is insufficient to describe a spin-1/2 particle interaction with gravity. However, for a weak field described by $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \ll 1$, a gauge invariant gravitational phase

$$\begin{aligned}\Phi_G &\equiv \frac{1}{2} \int_{x_0^\mu}^{x^\mu} dz^\lambda h_{\lambda\alpha}(z) \mathbf{p}^\alpha \\ &\quad - \frac{1}{4} \int_{x_0^\mu}^{x^\mu} dz^\lambda [h_{\lambda\alpha,\beta}(z) - h_{\lambda\beta,\alpha}(z)] \mathbf{L}^{\alpha\beta}(z) \\ &= \int_{t_0}^t dt' (\nabla_t \Phi_G) + \int_{x_0}^x dx' (\nabla_x \Phi_G) \\ &\quad + \int_{y_0}^y dy' (\nabla_y \Phi_G) + \int_{z_0}^z dz' (\nabla_z \Phi_G) \quad (4)\end{aligned}$$

defined along some space-time trajectory $z^\mu = (t', x', y', z')$ leads to a weak-field solution of the covariant Klein-Gordon equation [9–12], where \mathbf{p}^α and $\mathbf{L}^{\alpha\beta}$ are the generators of linear and orbital angular momentum for a free particle. Use of (4) according to the phase transformation $\psi(x) \rightarrow \exp(i\Phi_G/\hbar) \psi(x)$ results in a new Dirac Hamiltonian $H = H_0 + H_{\Phi_G}$, where

$$H_{\Phi_G} = \boldsymbol{\alpha} \cdot (\nabla \Phi_G) + (\nabla_t \Phi_G) \quad (5)$$

is treated as a perturbation of H_0 , reproducing all terms that have been observed, derived, or predicted to exist [13] for a spin-1/2 particle in a gravitational field.

Neutrino Wave Packets.—To formulate the wave packet description, we begin with a wavefunction composed of a linear superposition of plane waves [14]

$$|\psi\rangle = \frac{1}{(2\pi)^{3/2}} \int d^3k \xi(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}} |U(\mathbf{k})\rangle, \quad (6)$$

where $e^{i\mathbf{k} \cdot \mathbf{r}} |U(\mathbf{k})\rangle$ is a normalized solution of the free-particle Dirac equation, and

$$\xi(\mathbf{k}) = \frac{1}{(\sqrt{2\pi} \sigma_p)^{3/2}} \exp \left[-\frac{(\mathbf{k} - \mathbf{k}_0)^2}{4 \sigma_p^2} \right] \quad (7)$$

is a Gaussian function in momentum space, of width σ_p and centroid \mathbf{k}_0 , with $k_0 = |\mathbf{k}_0|$. The matrix element due to (5) is then

$$\begin{aligned}\langle \psi(\mathbf{r}) | H_{\Phi_G} | \psi(\mathbf{r}) \rangle &= \frac{1}{(2\pi)^3} \int d^3r' d^3k d^3k' \xi(\mathbf{k}) \xi(\mathbf{k}') \\ &\quad \times \exp[i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}'] \langle U(\mathbf{k}') | H_{\Phi_G}(\mathbf{r}, \mathbf{r}') | U(\mathbf{k}) \rangle, \quad (8)\end{aligned}$$

where the integration is performed over all phase space in spherical co-ordinates, excluding the region occupied by the gravitational source. To evaluate (8) explicitly, we need to specify $|U(\mathbf{k})\rangle$. Assuming the Weyl representation [2] for the gamma matrices, it is understood that the Dirac four-spinor is

$$|U(\mathbf{k})\rangle^{\text{Dirac}} = |\nu_L\rangle + |\nu_R\rangle, \quad (9)$$

where $|\nu_{L(R)}\rangle$ is its left- (right)-handed chiral projection. In contrast, a Majorana four-spinor is identical to itself (up to some phase) under charge conjugation [2], where $|\chi^c\rangle = \pm |\chi\rangle$. Then

$$|U(\mathbf{k})\rangle^{\text{Maj.}} = \begin{cases} |W_1(\mathbf{k})\rangle^{\text{Maj.}} \equiv |\nu_L\rangle + |\nu_L^c\rangle, \\ |W_2(\mathbf{k})\rangle^{\text{Maj.}} \equiv |\nu_R\rangle - |\nu_R^c\rangle, \end{cases} \quad (10)$$

and $|W_{1(2)}^c\rangle^{\text{Maj.}} = \pm |W_{1(2)}\rangle^{\text{Maj.}}$.

After substituting (9) or (10) into (8), we proceed to evaluate the Dirac or Majorana matrix element with the Rayleigh plane wave expansion [15] in terms of spherical Bessel functions and spherical harmonics defined for both position and momentum space angles. It follows that the orthonormality conditions serve to truncate the series expansion, allowing a virtually *exact* evaluation of the matrix element. Generically, (8) is the sum of both spin-diagonal terms and spin-flip terms proportional to the Pauli spin matrices $\boldsymbol{\sigma}^j$. Since the neutrino's spin quantization axis is parallel to the direction of propagation, the helicity transition element is [14]

$$\begin{aligned}\langle \pm | \boldsymbol{\sigma} | \mp \rangle &= [\cos \theta \cos \varphi \pm i \sin \varphi] \hat{\mathbf{x}} \\ &\quad + [\cos \theta \sin \varphi \mp i \cos \varphi] \hat{\mathbf{y}} - \sin \theta \hat{\mathbf{z}}, \quad (11)\end{aligned}$$

where $|\pm\rangle$ are the two-component spinors which define positive (negative) helicity for the neutrino.

Dirac and Majorana Matrix Elements.—After performing a power series expansion of (8) with respect to $\bar{m} \equiv m/(\hbar k_0) \ll 1$, we present the main formal results. For the Dirac neutrino, the gravitational phase-induced matrix element is

$$\begin{aligned}\langle \psi(\mathbf{r}) | H_{\Phi_G} | \psi(\mathbf{r}) \rangle^{\text{Dirac}} &= \\ &(\hbar k_0) \left\{ \frac{M}{r} [C_0 + C_1 \bar{m} + C_2 \bar{m}^2] \right. \\ &\quad \left. + \frac{M\Omega R^2}{r^2} \sin \theta [D_0 + D_1 \bar{m} + D_2 \bar{m}^2] \right\}, \quad (12)\end{aligned}$$

where C_j and D_j are dimensionless functions of k_0 , R , r , and $q \equiv k_0/\sigma_p$, whose explicit expressions are shown in a much longer paper [16]. Some important details in (12) are as follows. First, C_j correspond to the spin-diagonal parts of (8) coupled to M/r , while D_j refer to the spin-flip parts coupled to $M\Omega R^2/r^2$. Second, the presence of $\sin \theta$ clearly indicates that only terms with the z -component of (11) survive the integration. The most obvious interpretation is that the gravitational source's rotation induces the helicity transition of the Dirac neutrino for propagation away from the axis of symmetry. That is, the off-diagonal metric terms in (1) resemble an inhomogeneous magnetic field generating the spin-flip of a particle. Third, there are terms *linear* in \bar{m} present in (12) from the fact that the normalization coefficient in $|U(\mathbf{k})\rangle^{\text{Dirac}}$ is $\sqrt{(E+m)/(2E)}$, where $E = \sqrt{p^2 + m^2}$, with important consequences to follow.

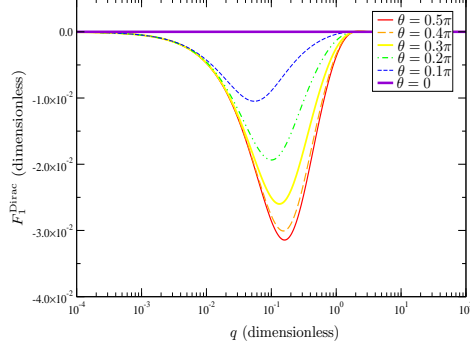
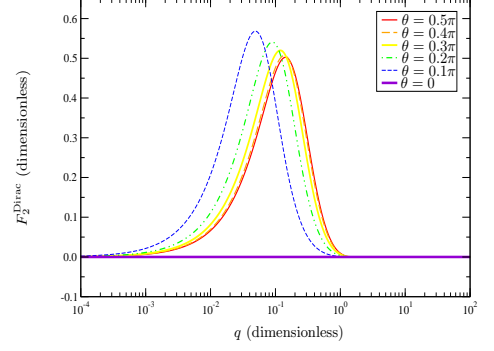
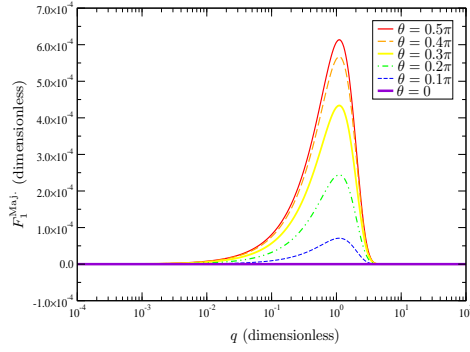
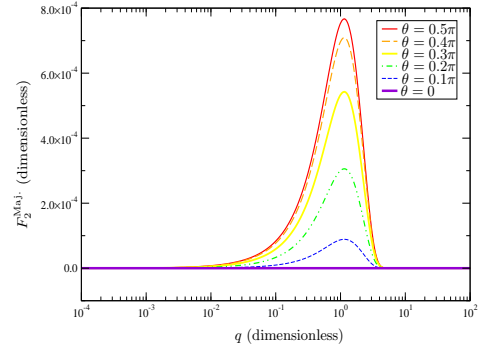
(a) F_1^{Dirac} (SN1987A)(a) F_2^{Dirac} (SN1987A)(b) $F_1^{\text{Maj.}}$ (SN1987A)(b) $F_2^{\text{Maj.}}$ (SN1987A)

FIG. 1: F_1 as a function of q due to the SN1987A gravitational source, for varying neutrino beam angle θ . Besides having opposite sign, F_1^{Dirac} is two orders of magnitude larger than $F_1^{\text{Maj.}}$.

Evaluating (8) for the Majorana neutrino, we have

$$\begin{aligned} \langle \psi_{1(2)}(\mathbf{r}) | H_{\Phi_G} | \psi_{1(2)}(\mathbf{r}) \rangle^{\text{Maj.}} = \\ (\hbar k_0) \left\{ \frac{M}{r} [C_0 + C_1 \bar{m} + C_2 \bar{m}^2] \right. \\ \left. \pm \sin \theta \sin \varphi \left[\frac{M}{r} \langle \pm | \boldsymbol{\sigma} | \mp \rangle^{\hat{y}} [C_{0\hat{y}} + C_{1\hat{y}} \bar{m} + C_{2\hat{y}} \bar{m}^2] \right] \right. \\ \left. + \frac{M \Omega R^2}{r^2} \langle \pm | \boldsymbol{\sigma} | \mp \rangle^{\hat{x}} [D_{0\hat{x}} + D_{1\hat{x}} \bar{m} + D_{2\hat{x}} \bar{m}^2] \right\}, \quad (13) \end{aligned}$$

where the “1(2)” refers to the upper (lower) signs in (13), and $C_{j\hat{y}}$ and $D_{j\hat{x}}$ are also dimensionless functions of k_0 , R , r , and q . The differences between (12) and (13) are quite striking. First, while both have the spin-diagonal terms C_j , the spin-flip terms have an overall factor of $\sin \theta \sin \varphi$. This corresponds to the y -component of the neutrino beam, a direct consequence of the Majorana neutrino’s self-conjugation condition, as the charge conjugation operation involves the presence of $\boldsymbol{\sigma}^{\hat{y}}$ in its definition [2]. This result suggests a preferred direction or-

FIG. 2: F_2 as a function of q applied to SN1987A for varying θ . The gravitational correction for F_2^{Dirac} is larger than $F_2^{\text{Maj.}}$ by three orders of magnitude.

thogonal to the source’s axis of symmetry. However, since the LT metric is *axisymmetric*, this φ -dependence on (13) purely results from how we defined the co-ordinate system beforehand. This anisotropy can be removed by averaging over a complete cycle. Even so, if the source has a significant quadrupole moment to induce an azimuthal perturbation of the LT metric, then a resonance effect may be generated under suitable conditions. Second, the spin-flip parts of (13) are dependent on the x - and y -components of (11), as opposed to the z -component for the Dirac neutrino. Third, a spin-flip term still contributes to the Majorana matrix element in the limit as $\Omega \rightarrow 0$, while no such term survives for the Dirac counterpart. Though this seems counterintuitive, terms of this type are expected to be present because of the self-conjugate nature of Majorana neutrinos.

Spin-Gravity Corrections to Neutrino Oscillation Length.—We now demonstrate how the matrix elements (12) and (13) lead to predicted gravitational corrections in the neutrino oscillation length, defined as $L_{\text{osc.}} =$

$2\pi / (E_{\bar{m}_2}^{(\pm)} - E_{\bar{m}_1}^{(\pm)})$ [1, 2], where by convention we set $m_2 > m_1$. To obtain $E_{\bar{m}_j}^{(\pm)}$, we use the Brillouin-Wigner (BW) method [17], instead of the more familiar Rayleigh-Schrödinger (RS) method [18]. From the BW approach applied to a second-order perturbation, we have

$$E_{\bar{m}_j}^{(\pm)} = E_0^{(\pm)} + \langle \pm | H_{\Phi_G} | \pm \rangle + \frac{|\langle \mp | H_{\Phi_G} | \pm \rangle|^2}{E_{\bar{m}_j}^{(\pm)} - E_0^{(\mp)}}, \quad (14)$$

where the unperturbed energy $E_0^{(\pm)}$ is described by (3), and $\langle \pm | H_{\Phi_G} | \pm \rangle$ and $\langle \mp | H_{\Phi_G} | \pm \rangle$ are the spin-diagonal and spin-flip components of the Dirac and Majorana matrix elements, as found in (12) and (13), respectively.

The advantage of the BW method comes from knowing that it yields an *exact* expression for the total energy involving a two-level spin system, which is precisely what we have. Therefore, we can solve for $E_{\bar{m}_2}^{(\pm)} - E_{\bar{m}_1}^{(\pm)}$ after averaging over the azimuthal angular dependence in the helicity transition term via

$$|\langle \mp | H_{\Phi_G} | \pm \rangle|^2 \rightarrow \frac{1}{2\pi} \int_0^{2\pi} |\langle \mp | H_{\Phi_G} | \pm \rangle|^2 d\varphi. \quad (15)$$

This leads to the expression

$$E_{\bar{m}_2}^{(\pm)} - E_{\bar{m}_1}^{(\pm)} = (\hbar k_0) \left[F_1 (\bar{m}_2 - \bar{m}_1) + \left(F_2 + \frac{1}{2} \right) (\bar{m}_2^2 - \bar{m}_1^2) \right], \quad (16)$$

where F_1 and F_2 are also dimensionless functions of k_0 , R , r , and q , and dependent on a complicated combination of terms coupled to M/r and $M\Omega R^2/r^2$, whose analytic expressions are deferred to the forthcoming longer paper [16]. As a test case, we present F_1 and F_2 as a function of q in Figures 1 and 2, respectively, for varying orientations of the neutrino beam angle θ , using data from SN1987A [19], where $M \approx 1.4 M_\odot$, $R \approx 10$ km, $\Omega \approx 2.936$ kHz, and $r \approx 49$ kpc. The neutrino wave packet is assumed to have a mean momentum of $\hbar k_0 = 1$ MeV throughout.

For all plots of F_j considered, the differences between positive and negative helicity are negligibly small. Also, the functions are insensitive to moderate or large changes in k_0 . Comparing Figures 1(a) and 1(b) for $\theta = \pi/2$, we see that F_1^{Dirac} and $F_1^{\text{Maj.}}$ have opposite sign, and $|F_1^{\text{Dirac}}|$ peaks near 3×10^{-2} , while the corresponding maximum $F_1^{\text{Maj.}}$ is roughly 6×10^{-4} . Regarding F_2 , Figures 2(a) and 2(b) for $\theta = \pi/2$ show that F_2^{Dirac} has a maximum value near 0.5, while $F_2^{\text{Maj.}}$ peaks near 8×10^{-4} , about three orders of magnitude smaller. The fact that F_j are non-zero for the range of $10^{-4} \lesssim q \lesssim 10^1$ is consistent with reasonable choices of σ_p for neutrinos produced in a neutron star, based on a mean free path calculation assuming known stellar data [14].

Conclusion.—This paper indicates that Dirac and Majorana wave packets interact differently with gravity, with

potential observational consequences. It is especially valuable to realize that the SN1987A values for F_1 , while small, are *not* negligible. This presents the interesting possibility that we can extract observational knowledge of the absolute mass difference $m_2 - m_1$ for a two-flavour oscillating system. Use of (16) can then lead to a determination of the absolute neutrino masses m_1 and m_2 by a parameter fit of q , $\bar{m}_2 - \bar{m}_1$, and $\bar{m}_2^2 - \bar{m}_1^2$ to precision measurements of the neutrino oscillation length, should this possibility become accessible in the future.

Acknowledgements.—This research is supported, in part, by the Natural Sciences and Engineering Research Council of Canada (NSERC).

* Electronic address: singhd@uregina.ca

† Electronic address: nader.mobed@uregina.ca

‡ Electronic address: papini@uregina.ca

- [1] R.N. Mohapatra and P.B. Pal, *Massive Neutrinos in Physics and Astrophysics*, 3rd Edition, (World Scientific 2004).
- [2] M. Fukugita and T. Yanagida, *Physics of Neutrinos and Applications to Astrophysics*, (Springer-Verlag Press 2003).
- [3] A. Zee, *Quantum Field Theory in a Nutshell*, (Princeton University Press 2003).
- [4] Y. Fukuda et al. (Superkamiokande Collaboration), *Phys. Rev. Lett.* **86**, 5656-5660 (2001); **86**, 5651-5655 (2001); **82**, 2644 (1999); *Phys. Lett. B* **433**, 9 (1998); **436**, 33 (1998); **467**, 185 (1999).
- [5] Q.R. Ahmad et al. (SNO Collaboration), *Phys. Rev. Lett.* **89**, 011302 (2002); **89**, 011301 (2002); **87**, 071301 (2001).
- [6] S.M. Bilenky, C. Giunti, and W. Grimus, *Prog. Part. Nucl. Phys.* **43**, 1 (1999).
- [7] J. Lense and H. Thirring, *Z. Phys.* **19**, 156 (1918); (English translation: B. Mashhoon, F.W. Hehl, and D.S. Theiss, *Gen. Rel. Grav.* **16**, 711 (1984)).
- [8] C.S. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation*, (W. H. Freeman & Co. 1973).
- [9] Y.Q. Cai and G. Papini, *Phys. Rev. Lett.* **66**, 1259 (1991); **68**, 3811 (1992).
- [10] D. Singh and G. Papini, *Nuovo Cimento* **B115**, 223 (2000).
- [11] G. Papini, in *Advances in the Interplay between Quantum and Gravity Physics*, edited by P.G. Bergmann and V. de Sabbata, 317 (Kluwer Academic 2002); arXiv:gr-qc/0110056.
- [12] G. Papini in *Relativity in Rotating Frames*, edited by G. Rizzi and M.L. Ruggiero (Kluwer Academic 2004) Ch. 16 and references therein; arXiv:gr-qc/0304082.
- [13] F.W. Hehl and W.-T. Ni, *Phys. Rev.* **D42**, 2045 (1990).
- [14] D. Singh, N. Mobed, and G. Papini, *Phys. Lett. A* **351**, 373 (2006).
- [15] G.B. Arfken and H.J. Weber, *Mathematical Methods for Physicists*, 4th Edition, (Academic Press 1995).
- [16] D. Singh, N. Mobed, and G. Papini, in progress.
- [17] L.E. Ballentine, *Quantum Mechanics*, (Prentice-Hall 1990)
- [18] J.J. Sakurai, *Modern Quantum Mechanics: Revised Edition*, (Addison-Wesley 1994).

- [19] K. Saida et al., *Class. Quantum Grav.* **20**, S645 (2003).